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Abstract

In this work it is presented an ADER-WENO approach for hyperbolic problems in the context of the finite volume method, using adaptive mesh refinement. ADER approach is of great interest for time integration since it achieves an arbitrary order of accuracy in a single time step. This is a joint work with M. Dumbser and O. Zanotti from the University of Trento (Italy).

Keywords: ADER approach, Adaptive Mesh Refinement, high order finite volume schemes.

Mathematics Subject Classifications (2000): 65Z05, 65M60

ADER-WENO Finite Volume Schemes with Adaptive Mesh Refinement for Hyperbolic Problems

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Abstract

A class of high order finite volume schemes for non-conservative problems is presented in this work. The numerical scheme is based on finite volume scheme using one-step arbitrary order ADER approach for time integration, using the local space-time discontinuous Galerkin approach, and high order WENO reconstruction in space. **Keywords:** climate models, high order finite volume scheme

Mathematics Subject Classifications (2000): 65Z05, 65M60

1 Introduction

Many problems related to fluid dynamics can be represented by systems of hyperbolic conservation laws. These appear when studying multiphase flows, shallow water models, relativistic problems among many others. Very often, different temporal and spatial scales are involved (this happens for instance in astrophysics). In these situations Adaptive Mesh Refinement (AMR), dynamic change of the computational grid, becomes necessary.

2 Mathematical model

We consider the 3D-hyperbolic system of balance laws in Cartesian coordinates

$$(1) \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} + \frac{\partial \mathbf{h}}{\partial z} = \mathbf{S}(\mathbf{u}, \mathbf{x}, t)$$

and carry out a finite volume discretization over the space-time control volume $\mathcal{T}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \times [z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}] \times [t^n, t^{n+1}]$ to yield

$$(2) \quad u_{ijk}^{n+1} = u_{ijk}^n - \frac{1}{\Delta x_i} \left(\mathbf{f}_{i+\frac{1}{2},j,k} - \mathbf{f}_{i-\frac{1}{2},j,k} \right) - \frac{1}{\Delta y_j} \left(\mathbf{g}_{i,j+\frac{1}{2},k} - \mathbf{g}_{i,j-\frac{1}{2},k} \right)$$

$$- \frac{1}{\Delta z_k} \left(\mathbf{h}_{i,j,k+\frac{1}{2}} - \mathbf{h}_{i,j,k-\frac{1}{2}} \right) + \Delta t \bar{\mathbf{S}}_{ijk}$$

where

$$(3) \quad u_{ijk}^n = \int_{S_i} \mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}^n) dS$$

where we have introduced the notation $\mathcal{S}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \times [z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}]$.

$$(4) \quad \mathbf{f}_{i+\frac{1}{2},j,k} = \frac{1}{\Delta t \Delta y \Delta z} \int_{t^n}^{t^{n+1}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \tilde{\mathbf{f}}(\mathbf{q}(x_{i-\frac{1}{2}}, y, z)) dy dz dt$$

$$(5) \quad \mathbf{g}_{i,j+\frac{1}{2},k} = \frac{1}{\Delta t \Delta x \Delta z} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \tilde{\mathbf{g}}(\mathbf{q}(x, y_{j-\frac{1}{2}}, z)) dz dx dt$$

$$(6) \quad \mathbf{h}_{i,j,k+\frac{1}{2}} = \frac{1}{\Delta t \Delta x \Delta y} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \tilde{\mathbf{h}}(\mathbf{q}(x, y, z_{k-\frac{1}{2}})) dy dx dt$$

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where \mathbf{q} is a space-time predictor. In order to reconstruct the numerical fluxes a local Lax-Friedrichs flux can be applied

$$(7) \quad \tilde{\mathbf{f}}(\mathbf{q}^-, \mathbf{q}^+) = \frac{1}{2}(\mathbf{f}(\mathbf{q}^+) + \mathbf{f}(\mathbf{q}^-)) + \frac{1}{2}|s_{\max}|(\mathbf{q}^+ - \mathbf{q}^-),$$

where $|s_{\max}|$ denotes the maximum eigenvalue of the Jacobian Matrix $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$, although other options are possible.

The main ingredients of this formulation are high-order reconstruction of fluxes at cell interfaces, achieved using Weighted Essentially Non Oscillatory (WENO) reconstruction and high-order evolution in time, using ADER-LSTDG approach.

We introduce the following operators,

$$(8) \quad \langle f, g \rangle_{T_i} = \int_{t^n}^{t^{n+1}} \int_{T_i} f(\vec{x}, t) g(\vec{x}, t) dV dt$$

$$(9) \quad [f, g]_{T_i}^t = \int_{T_i} f(\vec{x}, t) g(\vec{x}, t) dV$$

$$(10) \quad \{f, g\} = \int_{t^n}^{t^{n+1}} \int_{\partial T_i} f(\vec{x}, t) g(\vec{x}, t) dS dt$$

which denote the scalar products of two functions f and g over the space-time element $T_i \times [t^n; t^{n+1}]$, the spatial element T_i , and the space-time boundary element $\partial T_i \times [t^n; t^{n+1}]$ respectively. If we now multiply (??) with space-time basis functions Φ_k we obtain

In the context

$$(11) \quad \begin{aligned} & [\Phi_k, \mathbf{u}_h]_{T_i}^{t^{n+1}} + \langle \Phi_k, \mathbf{A}(\mathbf{W}_h) \cdot \nabla \mathbf{W}_h \rangle_{T_i \setminus \partial T_i} + \\ & + \langle \Phi_k, \nabla \mathbf{F} \rangle + \{ \Phi_k, \mathcal{D}(\mathbf{W}_h^-, \mathbf{W}_h^+, \mathbf{n}) \} \\ & = \langle \Phi_k, \mathbf{S}(\mathbf{W}_h) \rangle_{T_i}, \end{aligned}$$

where, \mathbf{u}_h are piecewise polynomials of degree N , W_h is the numerical solution of the space-time Galerkin predictor method introduced in ..., \mathbf{W}_h^- and \mathbf{W}_h^+ are the boundary extrapolated values of the element T_i and its neighbour, respectively, while $\mathcal{D}(\mathbf{W}_h^-, \mathbf{W}_h^+, \mathbf{n})$ denotes the jump term.

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